Numerical Analysis | (10th Edition)

Chapter 4.1, Problem 8E

Problem

The data in Exercise 6 were taken from the following functions. Compute the actual errors in Exercise 6, and find error bounds using the error formulas.

a. $f(x)=e^{2x}-\cos 2x$ **b.** $f(x)=\ln(x+2)-(x+1)^2$
c. $f(x)=x\sin x+x^2\cos x$ **d.** $f(x)=(\cos 3x)^2-e^{2x}$

Reference: Exercise 6

Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
−0.3	−0.27652	
−0.2	−0.25074	
−0.1	−0.16134	
0	0	

b.

x	$f(x)$	$f'(x)$
7.4	−68.3193	
7.6	−71.6982	
7.8	−75.1576	
8.0	−78.6974	

c.

x	$f(x)$	$f'(x)$
1.1	1.52918	
1.2	1.64024	
1.3	1.70470	
1.4	1.71277	

d.

x	$f(x)$	$f'(x)$
−2.7	0.054797	
−2.5	0.11342	
−2.3	0.65536	
−2.1	0.98472	

Step-by-step solution

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(a)

The data needed to estimate the first order derivative at the respective points using the most accurate three-point formula are as under

x	$f(x)$
-0.3	-0.27652
-0.2	-0.25074
-0.1	-0.16134
0	0

Compute the forward difference at $x_0=-0.3$ with $h=0.1$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h},\text{ so}$$

$$\begin{aligned}f'(-0.3)&=\frac{-3f(-0.3)+4f(-0.3+0.1)-f(-0.3+2\times0.1)}{2\times0.1}\\&=\frac{-3f(-0.3)+4f(-0.2)-f(-0.1)}{0.2}\\&=\frac{-3\times(-0.27652)+4\times(-0.25074)-(-0.16134)}{0.2}\end{aligned}$$

$$f'(-0.3)=-0.060300$$

Comment

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The truncation error in approximating the first order derivative (either backward or forward difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|,\quad \xi\in[x_0\ \ x_0+2h]$$

And the bound on this error is given as

$$\begin{aligned}\left|\frac{h^2}{3}f^{(3)}(\xi)\right|&\le\frac{h^2}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{(0.1)^2}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{0.01}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\end{aligned}$$

Here, $f(x)=e^{2x}-\cos(2x)$, therefore its first, second, and third derivatives are given as

$$f^{(1)}(x)=2\left[e^{2x}-\sin(2x)\right],$$
$$f^{(2)}(x)=4\left[e^{2x}+\cos(2x)\right],\text{ and}$$
$$f^{(3)}(x)=8\left[e^{2x}-\sin(2x)\right]$$

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{0.01}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)$$

Since $e^{2x}-\sin(2x)$ is a decreasing function in the interval $[-0.3\ \ 0.0]$ therefore

$$\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8\left[e^{2(-0.3)}-\sin\{2(-0.3)\}\right]$$
$$\Rightarrow\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8.9076$$

Thus, the error bound associated with the approximation of derivative at $x_0=-0.3$ is

$$\frac{0.01}{3}\left(\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=\frac{0.01}{3}\times8.9076$$
$$\frac{0.01}{3}\left(\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=2.9692\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=-0.3$ is

$$\begin{aligned}\left|-0.0603-2\left[e^{2(-0.3)}-\sin\{2(-0.3)\}\right]\right|&=\left|-0.0603-(-0.031661)\right|\\&=2.8639\times10^{-2}\end{aligned}$$

Comment

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Compute the central difference at $x_0=-0.2$ with $h=0.1$ of accuracy $O(h^2)$ using the midpoint formula

$$f'(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2h},\text{ so}$$

$$\begin{aligned}f'(-0.2)&=\frac{f(-0.2+0.1)-f(-0.2-0.1)}{2\times0.1}\\&=\frac{f(-0.1)-f(-0.3)}{0.2}\\&=\frac{-0.16134-(-0.27652)}{0.2}\end{aligned}$$

$$f'(-0.2)=0.57590$$

likewise, for $x_0=-0.1$ with $h=0.1$

$$\begin{aligned}f'(-0.1)&=\frac{f(-0.1+0.1)-f(-0.1-0.1)}{2\times0.1}\\&=\frac{f(0.0)-f(-0.2)}{0.2}\\&=\frac{0-(-0.25074)}{0.2}\end{aligned}$$

$$f'(-0.1)=1.2537$$

Comment

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Comment

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The truncation error in approximating the first order derivative (central difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{6}f^{(3)}(\xi)\right|,\quad \xi\in[x_0-h\ \ x_0+h]$$

And the bound on this error is given as

$$\begin{aligned}\left|\frac{h^2}{6}f^{(3)}(\xi)\right|&\le\frac{h^2}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{(0.1)^2}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{0.01}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\end{aligned}$$

Since $e^{2x}-\sin(2x)$ is a decreasing function in the interval $[-0.3\ \ 0.0]$ therefore

$$\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8\left[e^{2(-0.3)}-\sin\{2(-0.3)\}\right]$$
$$\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8.9076$$

Thus, the error bound associated with the approximation of derivative at $x_0=-0.2$ is

$$\frac{0.01}{6}\left(\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=\frac{0.01}{6}\times8.9076$$
$$\frac{0.01}{6}\left(\max_{\xi\in[-0.3\ -0.1]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=1.4846\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=-0.2$ is

$$\begin{aligned}\left|-0.5759-2\left[e^{2(-0.2)}-\sin\{2(-0.2)\}\right]\right|&=\left|-0.5759-0.5618\right|\\&=1.41\times10^{-2}\end{aligned}$$

Comment

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Again, since $e^{2x}-\sin(2x)$ is a decreasing function in the interval $[-0.3\ \ 0.0]$ therefore

$$\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8\left[e^{2(-0.2)}-\sin\{2(-0.2)\}\right]$$
$$\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8.4779$$

Thus, the error bound associated with the approximation of derivative at $x_0=-0.1$ is

$$\frac{0.01}{6}\left(\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=\frac{0.01}{6}\times8.4779$$
$$\frac{0.01}{6}\left(\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=1.4130\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=-0.1$ is

$$\begin{aligned}\left|1.2537-2\left[e^{2(-0.2)}-\sin\{2(-0.1)\}\right]\right|&=\left|1.2537-1.2401\right|\\&=1.36\times10^{-2}\end{aligned}$$

Comment

Step 7 of 38

Compute the backward difference at $x_0=0.0$ with $h=0.1$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{3f(x_0)-4f(x_0-h)+f(x_0+2h)}{2h},\text{ so}$$

$$\begin{aligned}f'(0.0)&=\frac{3f(0.0)-4f(0.0-0.1)+f(0.0-2\times0.1)}{2\times0.1}\\&=\frac{3f(0.0)-4f(-0.1)+f(-0.2)}{0.2}\\&=\frac{-3\times0-4\times(-0.16134)+(-0.25074)}{0.2}\end{aligned}$$

$$f'(0.0)=1.9731$$

Comment

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Since $e^{2x}-\sin(2x)$ is a decreasing function in the interval $[-0.3\ \ 0.0]$ therefore

$$\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8\left[e^{2(-0.2)}-\sin\{2(-0.2)\}\right]$$
$$\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|=8.4779$$

Thus, the error bound associated with the approximation of derivative at $x_0=-0.0$ is

$$\frac{0.01}{3}\left(\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=\frac{0.01}{3}\times8.4779$$
$$\frac{0.01}{3}\left(\max_{\xi\in[-0.2\ 0.0]}\left|8\left[e^{2\xi}-\sin(2\xi)\right]\right|\right)=2.826\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=-0.0$ is

$$\begin{aligned}\left|1.9731-2\left[e^{2(0.0)}-\sin\{2(-0.0)\}\right]\right|&=\left|1.9731-2\right|\\&=2.69\times10^{-2}\end{aligned}$$

Comment

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Hence, the estimate of the first order derivative at the tabulated points using the most accurate three points formula are as under

x	$f(x)$	$f'(x)$	actual error	error bounds
-0.3	-0.27652	-0.06030	2.86×10^{-2}	2.9692×10^{-2}
-0.2	-0.25074	0.57590	1.41×10^{-2}	1.4846×10^{-2}
-0.1	-0.16134	1.2537	1.36×10^{-2}	1.4130×10^{-2}
0	0	1.9731	2.69×10^{-2}	2.8260×10^{-2}

Comment

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Comment

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(b)

The data needed to estimate the first order derivative at the respective points using the most accurate three-point formula are as under

x	$f(x)$
7.4	-68.3193
7.6	-71.6982
7.8	-75.1576
8.0	-78.6974

Compute the forward difference at $x_0=7.4$ with $h=0.2$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h},\text{ so}$$

$$\begin{aligned}f'(7.4)&=\frac{-3f(7.4)+4f(7.4+0.2)-f(7.4+2\times0.2)}{2\times0.2}\\&=\frac{-3f(7.4)+4f(7.6)-f(7.8)}{0.4}\\&=\frac{-3\times(-68.3193)+4\times(-71.6982)-(-75.1576)}{0.4}\end{aligned}$$

$$f'(7.4)=-16.6932$$

Comment

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The truncation error in approximating the first order derivative (either backward or forward difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|,\quad \xi\in[x_0\ \ x_0+2h]$$

And the bound on this error is given as

$$\begin{aligned}\left|\frac{h^2}{3}f^{(3)}(\xi)\right|&\le\frac{h^2}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{(0.2)^2}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{0.04}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)\end{aligned}$$

Here, $f(x)=\ln(x+2)-(x+1)^2$, therefore its first, second, and third derivatives are given as

$$f^{(1)}(x)=\frac{1}{x+2}-2(x+1),$$
$$f^{(2)}(x)=-\frac{1}{(x+2)^2}-2,\text{ and}$$
$$f^{(3)}(x)=\frac{2}{(x+2)^3}$$

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{0.04}{3}\left(\max_{\xi\in[x_0\ x_0+2h]}\left|\frac{2}{(x+2)^3}\right|\right)$$

Since $\frac{2}{(x+2)^3}$ is a decreasing function in the interval $[7.4\ \ 8.0]$ therefore

$$\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|=\frac{2}{(7.4+2)^3}$$
$$\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|=0.00455606$$

Thus, the error bound associated with the approximation of derivative at $x_0=7.4$ is

$$\frac{0.04}{3}\left(\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|\right)=\frac{0.04}{3}\times0.00455606$$
$$\frac{0.04}{3}\left(\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|\right)=6.07475\times10^{-5}$$

And the actual error associated with the approximation of derivative at $x_0=7.4$ is

$$\begin{aligned}\left|-16.6932-\left[\frac{1}{7.4+2}-2(7.4+1)\right]\right|&=\left|-16.6932-(-16.6936)\right|\\&=4\times10^{-4}\end{aligned}$$

Comments (1)

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Compute the central difference at $x_0=7.6$ with $h=0.2$ of accuracy $O(h^2)$ using the midpoint formula

$$f'(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2h},\text{ so}$$

$$\begin{aligned}f'(7.6)&=\frac{f(7.6+0.2)-f(7.6-0.2)}{2\times0.2}\\&=\frac{f(7.8)-f(7.4)}{0.4}\\&=\frac{-75.1576-(-68.3193)}{0.4}\end{aligned}$$

$$f'(7.6)=-17.09575$$

likewise, for $x_0=7.8$ with $h=0.2$

$$\begin{aligned}f'(7.8)&=\frac{f(7.8+0.2)-f(7.8-0.2)}{2\times0.2}\\&=\frac{f(8.0)-f(7.6)}{0.4}\\&=\frac{-78.6974-(-71.6982)}{0.4}\end{aligned}$$

$$f'(7.8)=-17.498$$

Comment

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The truncation error in approximating the first order derivative (central difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{6}f^{(3)}(\xi)\right|,\quad \xi\in[x_0-h\ \ x_0+h]$$

And the bound on this error is given as

$$\begin{aligned}\left|\frac{h^2}{6}f^{(3)}(\xi)\right|&\le\frac{h^2}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{(0.2)^2}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\\&\le\frac{0.04}{6}\left(\max_{\xi\in[x_0-h\ x_0+h]}\left|f^{(3)}(\xi)\right|\right)\end{aligned}$$

Since $\frac{2}{(x+2)^3}$ is a decreasing function in the interval $[7.4\ \ 8.0]$ therefore

$$\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|=\frac{2}{(7.4+2)^3}$$
$$\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|=0.00455606$$

Thus, the error bound associated with the approximation of derivative at $x_0=7.6$ is

$$\frac{0.04}{6}\left(\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|\right)=\frac{0.04}{6}\times0.00455606$$
$$\frac{0.04}{6}\left(\max_{\xi\in[7.4\ 7.8]}\left|\frac{2}{(x+2)^3}\right|\right)=3.0374\times10^{-5}$$

And the actual error associated with the approximation of derivative at $x_0=7.6$ is

$$\begin{aligned}\left|-17.09575-\left[\frac{1}{7.6+2}-2(7.6+1)\right]\right|&=\left|-17.09575-(-17.09583)\right|\\&=8\times10^{-5}\end{aligned}$$

Comment

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$$\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|=\frac{2}{(7.6+2)^3}$$
$$\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|=0.00226056$$

Thus, the error bound associated with the approximation of derivative at $x_0=7.8$ is

$$\frac{0.04}{6}\left(\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|\right)=\frac{0.04}{6}\times 0.00226056$$
$$\frac{0.04}{6}\left(\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|\right)=1.50704\times10^{-5}$$

And the actual error associated with the approximation of derivative at $x_0=7.8$ is

$$\left|-17.498-\left[\frac{1}{7.8+2}-2(7.8+1)\right]\right|=\left|-17.498-(-17.4980)\right|$$
$$=0$$

Comment

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Compute the backward difference at $x_0=8.0$ with $h=0.2$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{3f(x_0)-4f(x_0-h)+f(x_0-2h)}{2h},\ \text{so}$$
$$f'(8.0)=\frac{3f(8.0)-4f(8.0-0.2)+f(8.0-2\times0.2)}{2\times0.1}$$
$$=\frac{3f(8.0)-4f(7.8)+f(7.6)}{0.2}$$
$$=\frac{-3\times(-78.6974)-4\times(-75.1576)+(-71.6982)}{0.2}$$
$$f'(8.0)=-17.9$$

Comment

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Since $\frac{2}{(x+2)^3}$ is a decreasing function in the interval $[7.4\ \ 8.0]$ therefore

$$\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|=\frac{2}{(7.6+2)^3}$$
$$\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|=0.00226056$$

Thus, the error bound associated with the approximation of derivative at $x_0=8.0$ is

$$\frac{0.04}{3}\left(\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|\right)=\frac{0.04}{3}\times 0.00226056$$
$$\frac{0.04}{3}\left(\max_{\xi\in[7.6\ \ 8.0]}\left|\frac{2}{(x+2)^3}\right|\right)=3.01408\times10^{-5}$$

And the actual error associated with the approximation of derivative at $x_0=8.0$ is

$$\left|-17.92-\left[\frac{1}{8.0+2}-2(8.0+1)\right]\right|=\left|-17.9-(-17.9)\right|$$
$$=0$$

Comment

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Comment

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Hence, the estimate of the first order derivative at the tabulated points using the most accurate three points formula are as under

x	$f(x)$	$f'(x)$	actual error	error bounds
7.4	-68.3193	-16.6932	4×10^{-4}	6.07475×10^{-5}
7.6	-71.6982	-17.0958	8×10^{-5}	3.03740×10^{-5}
7.8	-75.1576	-17.4980	0	1.50704×10^{-5}
8.0	-78.6974	-17.9000	0	3.01408×10^{-5}

Comment

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(c)

The data needed to estimate the first order derivative at the respective points using the most accurate three-point formula are as under

x	$f(x)$
1.1	1.52918
1.2	1.64024
1.3	1.70470
1.4	1.71277

Compute the forward difference at $x_0=1.1$ with $h=0.1$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h},\ \text{so}$$
$$f'(1.1)=\frac{-3f(1.1)+4f(1.1+0.1)-f(1.1+2\times0.1)}{2\times0.1}$$
$$=\frac{-3f(1.1)+4f(1.2)-f(1.3)}{0.2}$$
$$=\frac{-3\times(1.52918)+4\times(1,64024)-1.70470}{0.2}$$
$$f'(1.1)=1.3436$$

Comment

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The truncation error in approximating the first order derivative (either backward or forward difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|,\qquad \xi\in[x_0\ \ x_0+2h]$$

And the bound on this error is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{h^2}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{(0.1)^2}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{0.01}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$

Here, $f(x)=x\sin(x)+x^2\cos(x)$, therefore its first, second, and third derivatives are given as

$$f^{(1)}(x)=-x^2\sin(x)+\sin(x)+3x\cos(x),$$
$$f^{(2)}(x)=-(x^2-4)\cos(x)-5x\sin(x),\ \text{and}$$
$$f^{(3)}(x)=(x^2-9)\sin(x)-7x\cos(x)$$
$$\therefore\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{0.01}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)$$

Since $\left|(x^2-9)\sin(x)-7x\cos(x)\right|$ is a decreasing function in the interval $[1.1\ \ 1.4]$ therefore

$$\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=\left|(1.1^2-9)\sin(1.1)-7(1.1)\cos(1.1)\right|$$
$$\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=10.4352$$

Thus, the error bound associated with the approximation of derivative at $x_0=1.1$ is

$$\frac{0.01}{3}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=\frac{0.01}{3}\times10.4352$$
$$\frac{0.01}{3}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=3.4784\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=1.1$ is

$$\left|1.3436-\left[-(1.1)^2\sin(1.1)+\sin(1.1)+3(1.1)\cos(1.1)\right]\right|=\left|1.3436-1.3097\right|$$
$$=3.39\times10^{-2}$$

Comment

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Compute the central difference at $x_0=1.2$ with $h=0.1$ of accuracy $O(h^2)$ using the midpoint formula

$$f'(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2h},\ \text{so}$$
$$f'(1.2)=\frac{f(1.1+0.1)-f(1.1-0.1)}{2\times0.1}$$
$$=\frac{f(1.2)-f(1.0)}{0.2}$$
$$=\frac{1.70470-1.52918}{0.2}$$
$$f'(1.2)=0.8776$$

likewise, for $x_0=1.3$ with $h=0.1$

$$f'(1.3)=\frac{f(1.3+0.1)-f(1.3-0.1)}{2\times0.1}$$
$$=\frac{f(1.4)-f(1.2)}{0.2}$$
$$=\frac{1.71277-1.64024}{0.2}$$
$$f'(1.3)=0.36265$$

Comment

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The truncation error in approximating the first order derivative (central difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{6}f^{(3)}(\xi)\right|,\qquad \xi\in[x_0-h\ \ x_0+h]$$

And the bound on this error is given as

$$\left|\frac{h^2}{6}f^{(3)}(\xi)\right|\le\frac{h^2}{6}\left(\max_{\xi\in[x_0-h\ \ x_0+h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{(0.1)^2}{6}\left(\max_{\xi\in[x_0-h\ \ x_0+h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{0.01}{6}\left(\max_{\xi\in[x_0-h\ \ x_0+h]}\left|f^{(3)}(\xi)\right|\right)$$

Since $\left|(x^2-9)\sin(x)-7x\cos(x)\right|$ is a decreasing function in the interval $[1.1\ \ 1.4]$ therefore

$$\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=\left|(1.1^2-9)\sin(1.1)-7(1.1)\cos(1.1)\right|$$
$$\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=10.4352$$

Thus, the error bound associated with the approximation of derivative at $x_0=1.2$ is

$$\frac{0.01}{6}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=\frac{0.01}{6}\times10.4352$$
$$\frac{0.01}{6}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=1.7392\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=1.2$ is

$$\left|0.8776-\left[-(1.2)^2\sin(1.2)+\sin(1.2)+3(1.2)\cos(1.2)\right]\right|=\left|0.8776-0.6031\right|$$
$$=2.74\times10^{-1}$$

Comment

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Again, since $\left|(x^2-9)\sin(x)-7x\cos(x)\right|$ is a decreasing function in the interval $[1.1\ \ 1.4]$ therefore

$$\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=\left|(1.2^2-9)\sin(1.2)-7(1.2)\cos(1.2)\right|$$
$$\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=10.0900$$

Thus, the error bound associated with the approximation of derivative at $x_0=1.3$ is

$$\frac{0.01}{6}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=\frac{0.01}{6}\times10.0900$$
$$\frac{0.01}{6}\left(\max_{\xi\in[1.1\ \ 1.3]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=1.6817\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=1.3$ is

$$\left|0.36265-\left[-(1.3)^2\sin(1.3)+\sin(1.3)+3(1.3)\cos(1.3)\right]\right|=\left|0.36265-0.10839\right|$$
$$=2.5425\times10^{-1}$$

Comment

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Compute the backward difference at $x_0=1.4$ with $h=0.1$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{3f(x_0)-4f(x_0-h)+f(x_0-2h)}{2h},\ \text{so}$$
$$f'(1.4)=\frac{3f(1.4)-4f(1.4-0.1)+f(1.4-2\times0.1)}{2\times0.1}$$
$$=\frac{3f(1.4)-4f(1.3)+f(1.2)}{0.2}$$
$$=\frac{3\times1.71277+4\times1.70470-1.64024}{0.2}$$
$$f'(1.4)=-0.20125$$

Comment

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Since $\left|(x^2-9)\sin(x)-7x\cos(x)\right|$ is a decreasing function in the interval $[1.1\ \ 1.4]$ therefore

$$\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=\left|(1.2^2-9)\sin(1.2)-7(1.2)\cos(1.2)\right|$$
$$\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|=10.0900$$

Thus, the error bound associated with the approximation of derivative at $x_0=1.4$ is

$$\frac{0.01}{3}\left(\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=\frac{0.01}{3}\times10.0900$$
$$\frac{0.01}{3}\left(\max_{\xi\in[1.2\ \ 1.4]}\left|(x^2-9)\sin(x)-7x\cos(x)\right|\right)=3.36333\times10^{-2}$$

And the actual error associated with the approximation of derivative at $x_0=1.4$ is

$$\left|-0.20125-\left[-(1.4)^2\sin(1.4)+\sin(1.4)+3(1.4)\cos(1.4)\right]\right|=\left|-0.20125-(-0.232170)\right|$$
$$=3.092\times10^{-2}$$

Comment

Step 27 of 38

Hence, the estimate of the first order derivative at the tabulated points using the most accurate three points formula are as under

x	$f(x)$	$f'(x)$	actual error	error bounds
1.1	1.52918	1.34360	3.39×10^{-2}	3.4784×10^{-2}

Comment

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1.2

1.64024

0.877600

2.74×10^{-1}

1.7392×10^{-2}

1.3

1.70470

0.362650

2.5425×10^{-1}

1.6817×10^{-2}

1.4

1.71277

-0.201250

3.092×10^{-2}

3.36333×10^{-2}

Comment

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(d)

The data needed to estimate the first order derivative at the respective points using the most accurate three-point formula are as under

x	$f(x)$
-2.7	0.054797
-2.5	0.11342
-2.3	0.65536
-2.1	0.98472

Compute the forward difference at $x_0=-2.7$ with $h=0.2$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0)=\frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h},\ \text{so}$$
$$f'(2.7)=\frac{-3f(-2.7)+4f(-2.7+0.2)-f(-2.7+2\times0.2)}{2\times0.2}$$
$$=\frac{-3f(-2.7)+4f(-2.5)-f(-2.3)}{0.4}$$
$$=\frac{-3\times0.054797+4\times0.11342-0.65536}{0.4}$$
$$f'(-2.7)=-0.91518$$

Comment

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The truncation error in approximating the first order derivative (either backward or forward difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|,\qquad \xi\in[x_0\ \ x_0+2h]$$

And the bound on this error is given as

$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{h^2}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{(0.2)^2}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$
$$\le\frac{0.04}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|f^{(3)}(\xi)\right|\right)$$

Here, $f(x)=\{\cos(3x)\}^2-e^{2x}$, therefore its first, second, and third derivatives are given as

$$f^{(1)}(x)=-2\{e^{2x}+3\sin(3x)\cos(3x)\},$$
$$f^{(2)}(x)=-2\{2e^{2x}-9\sin^2(3x)+9\cos^2(3x)\},\ \text{and}$$
$$f^{(3)}(x)=-8\{e^{2x}-27\sin(3x)\cos(3x)\}$$
$$\left|\frac{h^2}{3}f^{(3)}(\xi)\right|\le\frac{0.04}{3}\left(\max_{\xi\in[x_0\ \ x_0+2h]}\left|-8\{e^{2x}-27\sin(3x)\cos(3x)\}\right|\right)$$

Since $\left|-8\{e^{2x}-27\sin(3x)\cos(3x)\}\right|$ is neither increasing nor decreasing function in the interval $[-2.7\ \ -2.1]$ and its maximum occurs at $x=-2.3562$

$$\max_{\xi\in[-2.7\ \ -2.3]}\left|-8\{e^{2\xi}-27\sin(3\xi)\cos(3\xi)\}\right|=\left|-8\left[e^{2(-2.3562)}-27\sin\{3(-2.3562)\}\cos\{3(-2.3562)\}\right]\right|$$
$$\max_{\xi\in[-2.7\ \ -2.3]}\left|-8\{e^{2\xi}-27\sin(3\xi)\cos(3\xi)\}\right|=108.05$$

Thus, the error bound associated with the approximation of derivative at $x_0=-2.7$ is

$$\frac{0.04}{3}\left(\max_{\xi\in[-2.7\ \ -2.3]}\left|-8\{e^{2\xi}-27\sin(3\xi)\cos(3\xi)\}\right|\right)=\frac{0.04}{3}\times108.05$$
$$\frac{0.04}{3}\left(\max_{\xi\in[-2.7\ \ -2.3]}\left|-8\{e^{2\xi}-27\sin(3\xi)\cos(3\xi)\}\right|\right)=1.4407$$

And the actual error associated with the approximation of derivative at $x_0=-2.7$ is

$$\left|-0.91518-\left[-2\left\{e^{2(-2.7)}+3\sin\{3(-2.7)\}\cos\{3(-2.7)\}\right\}\right]\right|=\left|-0.91518-(-1.4263)\right|$$
$$=5.1112\times10^{-1}$$

Comment

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Compute the central difference at $x_0=-2.5$ with $h=0.2$ of accuracy $O(h^2)$ using the midpoint formula

$$f'(x_0)=\frac{f(x_0+h)-f(x_0-h)}{2h},\ \text{so}$$

$$= \frac{0.65536 - 0.054797}{0.4}$$

$f'(-2.5) = 1.5014$

likewise, for $x_0 = -2.3$ with $h = 0.2$

$$\begin{aligned} f'(-2.3) &= \frac{f(-2.3+0.2) - f(-2.3-0.2)}{2\times 0.2} \\ &= \frac{f(-2.1) - f(-2.5)}{0.4} \\ &= \frac{0.98472 - 0.11342}{0.4} \end{aligned}$$

$f'(-2.3) = 2.1782$

Comment

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The truncation error in approximating the first order derivative (central difference) with accuracy $O(h^2)$ is given as

$$\left|\frac{h^2}{6} f^{(3)}(\xi)\right|, \qquad \xi \in [x_0 - h \;\; x_0 + h]$$

And the bound on this error is given as

$$\begin{aligned} \left|\frac{h^2}{6} f^{(3)}(\xi)\right| &\le \frac{h^2}{6}\left(\max_{\xi\in[x_0-h \;\; x_0+h]} \left|f^{(3)}(\xi)\right|\right) \\ &\le \frac{(0.2)^2}{6}\left(\max_{\xi\in[x_0-h \;\; x_0+h]} \left|f^{(3)}(\xi)\right|\right) \\ &\le \frac{0.04}{6}\left(\max_{\xi\in[x_0-h \;\; x_0+h]} \left|f^{(3)}(\xi)\right|\right) \end{aligned}$$

Since $\left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|$ is neither increasing nor decreasing function in the interval $[-2.7 \;\; -2.1]$ and its maximum occurs at $x = -2.3562$

$$\max_{\xi\in[-2.7 \; -2.3]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = \left|-8\left[e^{2(-2.3562)} - 27\sin\{3(-2.3562)\}\cos\{3(-2.3562)\}\right]\right|$$

$$\max_{\xi\in[-2.7 \; -2.3]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = 108.05$$

Thus, the error bound associated with the approximation of derivative at $x_0 = -2.5$ is

$$\frac{0.04}{6}\left(\max_{\xi\in[-2.7 \; -2.3]} \left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|\right) = \frac{0.04}{6}\times 108.05$$

$$\frac{0.04}{6}\left(\max_{\xi\in[-2.7 \; -2.3]} \left|8\left[e^{2\xi} - \sin(2\xi)\right]\right|\right) = 7.203\times 10^{-1}$$

And the actual error associated with the approximation of derivative at $x_0 = -2.5$ is

$$\begin{aligned} \left|1.5014 - \left[-2\left\{e^{2(-2.5)} + 3\sin\{3(-2.5)\}\cos\{3(-2.5)\}\right\}\right]\right| &= \left|1.5014 - 1.9373\right| \\ &= 4.359\times 10^{-1} \end{aligned}$$

Comment

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Again, since $\left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|$ is neither increasing nor decreasing function in the interval $[-2.7 \;\; -2.1]$ and its maximum occurs at $x = -2.3562$

$$\max_{\xi\in[-2.1 \; -2.5]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = \left|-8\left[e^{2(-2.3562)} - 27\sin\{3(-2.3562)\}\cos\{3(-2.3562)\}\right]\right|$$

$$\max_{\xi\in[-2.1 \; -2.5]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = 108.05$$

Thus, the error bound associated with the approximation of derivative at $x_0 = -2.3$ is

$$\frac{0.04}{6}\left(\max_{\xi\in[-2.1 \; -2.5]} \left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|\right) = \frac{0.04}{6}\times 108.05$$

$$\frac{0.04}{6}\left(\max_{\xi\in[-2.1 \; -2.5]} \left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|\right) = 7.203\times 10^{-1}$$

And the actual error associated with the approximation of derivative at $x_0 = -2.3$ is

$$\begin{aligned} \left|2.1782 - \left[-2\left\{e^{2(-2.3)} + 3\sin\{3(-2.3)\}\cos\{3(-2.3)\}\right\}\right]\right| &= \left|2.1782 - 2.8110\right| \\ &= 6.328\times 10^{-1} \end{aligned}$$

Comment

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Compute the backward difference at $x_0 = -2.1$ with $h = 0.2$ of accuracy $O(h^2)$ using endpoint formula

$$f'(x_0) = \frac{3f(x_0) - 4f(x_0 - h) + f(x_0 - 2h)}{2h}, \text{ so}$$

$$\begin{aligned} f'(-2.1) &= \frac{3f(-2.1) - 4f(-2.1-0.2) + f(-2.1-2\times 0.2)}{2\times 0.2} \\ &= \frac{3f(-2.1) - 4f(-2.3) + f(-2.5)}{0.4} \\ &= \frac{3\times 0.98472 - 4\times 0.65536 + 0.11342}{0.4} \end{aligned}$$

$f'(-2.1) = 1.1154$

Comment

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Since $\left|-8\{e^{2x} - 27\sin(3x)\cos(3x)\}\right|$ is neither increasing nor decreasing function in the interval $[-2.7 \;\; -2.1]$ and its maximum occurs at $x = -2.3562$

$$\max_{\xi\in[-2.5 \; -2.1]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = \left|-8\left[e^{2(-2.3562)} - 27\sin\{3(-2.3562)\}\cos\{3(-2.3562)\}\right]\right|$$

$$\max_{\xi\in[-2.5 \; -2.1]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right| = 108.05$$

Thus, the error bound associated with the approximation of derivative at $x_0 = -2.1$ is

$$\frac{0.04}{3}\left(\max_{\xi\in[-2.5 \; -2.1]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right|\right) = \frac{0.04}{3}\times 108.05$$

$$\frac{0.04}{3}\left(\max_{\xi\in[-2.5 \; -2.1]} \left|-8\{e^{2\xi} - 27\sin(3\xi)\cos(3\xi)\}\right|\right) = 1.4407$$

And the actual error associated with the approximation of derivative at $x_0 = -2.7$ is

$$\begin{aligned} \left|1.1154 - \left[-2\left\{e^{2(-2.1)} + 3\sin\{3(-2.1)\}\cos\{3(-2.1)\}\right\}\right]\right| &= \left|1.1154 - 0.070878\right| \\ &= 1.0445 \end{aligned}$$

Comment

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Hence, the estimate of the first order derivative at the tabulated points using the most accurate three points formula are as under

x	$f(x)$	$f'(x)$	actual error	error bounds
-2.7	0.054797	-0.91518	5.1112×10^{-1}	1.4407
-2.5	0.11342	1.5014	4.359×10^{-1}	7.203×10^{-1}
-2.3	0.65536	2.1782	6.328×10^{-1}	7.203×10^{-1}

Comment

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-2.1

0.98472

1.1154

1.0445

1.4407

Comment

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Comment